Effects of boundary reflection on radiative heat transfer in participating media

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Radiative heat transfer through a non-isothermal grey participating medium between two parallel surfaces kept at fixed temperature has been investigated. The integro-differential transfer equations for surface reflection were solved in semi-analytical form by projectional methods; conduction and convection were neglected. It was assumed that reflection from the cold wall was diffuse, while that from the hot wall was either diffuse or specular. The heat flux and the temperature distribution in the participating medium were calculated in each physical condition, in order to compare the effects of different reflection modes on heat transfer. The results show that temperature distributions and heat fluxes are only slightly affected by the particular reflection law, the relative difference being less than 1%. This suggests that diffuse reflection only could be considered for practical applications, since it requires a much simpler computational procedure.

Keywords: *radiative heat transfer, diffuse reflection, specular reflection, semianalytical solution*

Many engineering applications of radiative heat transfer involve the assumption of nonisothermal scattering, emitting and absorbing media confined within grey surfaces. Heat transport in nuclear reactors[1], porous materials, rocket performance, planetary re-entry, and industrial furnaces are examples where radiation is fundamental in determining the heat fluxes within the system.

Several review articles and papers can be found in the literature and many semianalytical[2-4] or numerical[5] methods, promoted by the availability of powerful computers, have been developed for particular situations. Moreover, there is currently a demand for greater accuracy in radiative transport predictions, to meet the rigorous safety and efficiency criteria required in applications of advanced heat transfer.

The equations rigorously describing the transport processes in the presence of a participating medium are easily derived, but difficulty arises when solving the governing integro-differential equation. The nature of the problem is such that, at any point of the system, the solution is influenced by processes occurring throughout the entire solution domain. Moreover, on all boundaries or interfaces, integral equations are again required in general to account for reflection of radiation.

The energy reflected by the surfaces is completely specular only if the surface is perfectly smooth. In engineering practice the surfaces are generally rough, and their irregularities may be of the order of magnitude of the wavelength of thermal radiation (a few microns). As a result, a monodirectional incident flux contributes to the reflected energy in every direction, and the global reflection is made up of specular and diffuse components. These features make tedious the elaboration of the radiative transport equations and complicate their solution.

In this paper the quantitative effects of the surface reflection are investigated through the determination of approximate analytical solutions, which may be made as accurate as is required, for two different problems: one relevant to diffuse reflection from both boundaries, the other to diffuse reflection from one wall, and specular reflection from the other. Aside from their intrinsic value, these semianalytical calculations also provide standard bench-mark solutions and evidence on the differing roles of diffuse and specular reflection laws.

Formulation

Consider an absorbing, emitting, isotropically scattering, nonisothermal grey medium of optical thickness $2a$ mean free paths, bounded by parallel walls (hot labelled 1 and cold labelled 2) of infinite lateral extent. The grey walls emit diffusely (Lambert's law) and are kept at two different fixed temperatures. Further, it is assumed that radiation is the only mode of energy transfer and the system is in steady state and in thermal equilibrium with purely diffuse reflection from the cold wall, on the right edge. Since

$$I_0(\tau)=2\pi\int_{-1}^{1} I(\tau,\mu)\,d\mu,\; q^{\pm}(\tau)=2\pi\int_0^1 \mu I(\tau,\pm\mu)\,d\mu \tag{1}$$

and with $\varepsilon_i+\rho_i^s+\rho_i^d=1$, $i=1,2$ (zero transmissivity of each wall) the transfer equation and boundary conditions

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read as

$$\mu\frac{\partial I}{\partial\tau}+I(\tau,\mu)=\frac{1}{4\pi}I_0(\tau)$$

$$I(-a,\mu)=\varepsilon_1\frac{n^2\sigma T_1^4}{\pi}+\rho_1^s I(-a,-\mu)+\frac{1}{\pi}\rho_1^d q^-(-a) \qquad \mu>0 \tag{2}$$

$$I(a,-\mu)=\varepsilon_2\frac{n^2\sigma T_2^4}{\pi}+\frac{1}{\pi}\rho_2^d q^+(a) \qquad \mu>0$$

where the constant heat flux q can be computed as q^+-q^-, and the temperature is given by

$$T^4(\tau)=\frac{1}{4n^2\sigma}I_0(\tau) \tag{3}$$

According to the general algorithm presented in Ref 6, and exploited later in a series of papers for negligible radiation emission by the medium itself[7-8], the transfer equation can be formally solved, to express $I(\tau,\mu)$ in terms of $I_0(\tau)$, $I(-a,-\mu)$, $q^-(-a)$ and $q^+(a)$, and then the boundary values $I(-a,-\mu)$, $q^-(-a)$ and $q^+(a)$ can be recomputed by suitable specializations and integrations of the above expression of $I(\tau,\mu)$, to yield a set of degenerate integral equations, which are analytically solvable and provide such boundary values as functions of I_0. The quite lengthy and tedious procedure is omitted here; the final result is that the radiation field $I(\tau,\mu)$ is completely determined by its first angular moment only, $I_0(\tau)$, in a relationship which accounts for all physical inputs and especially reflection from the boundaries.

In order to evaluate, then, the only unknown, I_0, it is sufficient to integrate $I(\tau,\mu)$ so that a linear integral Fredholm equation is obtained for $I_0(\tau)$, with both square summable kernel and inhomogeneous term. Since, in addition, the spectral radius of the integral operator is less than unity, provided one at least of the total reflectivities is not equal to one[9-10] (the opposite would correspond to the unphysical case of zero emissivity), the integral equation above has a unique square summable solution that is the limit in the norm of the sequence of approximate solutions obtained by any suitable projection method[11]. The results for the two extreme cases concerning reflection at the first wall are reported below.

a) For $\rho_1^s=0$ the symmetry of the problem allows considerable simplifications[12] and heat flux and temperature turn out to be

$$q=\frac{n^2\sigma(T_1^4-T_2^4)Q}{1+\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}-2\right)Q} \tag{4a}$$

$$\frac{T^4(\tau)-T_2^4}{T_1^4-T_2^4}=\frac{\Theta(\tau)+\left(\frac{1}{\varepsilon_2}-1\right)Q}{1+\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}-2\right)Q} \tag{4b}$$

where

$$Q=1-2\int_{-a}^{a}E_2(a+\tau)\Theta(\tau)\,d\tau \tag{5}$$

and $\Theta(\tau)$ is a universal dimensionless function, linearly related to $I_0(\tau)$, solution to the integral equation

$$\Theta(\tau)=\frac{1}{2}\int_{-a}^{a}E_1(|\tau-\tau'|)\Theta(\tau')\,d\tau'+\tfrac{1}{2}E_2(a+\tau) \tag{6}$$

b) For $\rho_1^d=0$ the integral equation for I_0 becomes

$$I_0(\tau)=\int_{-a}^{a}K(\tau,\tau')I_0(\tau')\,d\tau'+S(\tau) \tag{7}$$

with kernel

$$K(\tau,\tau')=\tfrac{1}{2}E_1(|\tau-\tau'|)+\tfrac{1}{2}(1-\varepsilon_1)E_1(2a+\tau+\tau')+\frac{1-\varepsilon_2}{1-2(1-\varepsilon_1)(1-\varepsilon_2)E_3(4a)}\times[E_2(a-\tau)+(1-\varepsilon_1)E_2(3a+\tau)]\times[E_2(a-\tau')+(1-\varepsilon_1)E_2(3a+\tau')] \tag{8a}$$

and inhomogeneous term

$$S(\tau)=2\varepsilon_1 n^2\sigma T_1^4E_2(a+\tau)+\frac{2n^2\sigma[2(1-\varepsilon_2)E_3(2a)\varepsilon_1T_1^4+\varepsilon_2T_2^4]}{1-2(1-\varepsilon_1)(1-\varepsilon_2)E_3(4a)}\times[E_2(a-\tau)+(1-\varepsilon_1)E_2(3a+\tau)] \tag{8b}$$

while $T(\tau)$ follows from Eq (3), and the heat flux is a

Notation

- a Optical half-thickness in mean free paths
- C Chebyshev polynomial of the first kind
- E_n nth exponential integral function
- I Angular radiation intensity
- n Refractive index
- P Legendre polynomial
- q Heat flux
- T Absolute temperature
- ε Emissivity
- μ Direction cosine
- ρ Reflectivity
- σ Stefan–Boltzmann constant
- τ Position measured from midpoint of participating material, ranging from $-a$ to $+a$

Superscripts

- $\pm$ Positive or negative direction
- s Specular
- d Diffuse

constant given by

$$q=\frac{1}{2}\int_{-a}^{a}\operatorname{sgn}(\tau-\tau')E_2(|\tau-\tau'|)I_0(\tau')\,d\tau'$$
$$+\tfrac{1}{2}(1-\varepsilon_1)\int_{-a}^{a}E_2(2a+\tau+\tau')I_0(\tau')\,d\tau'$$
$$-\frac{(1-\varepsilon_2)[E_3(a-\tau)-(1-\varepsilon_1)E_3(3a+\tau)]}{1-2(1-\varepsilon_1)(1-\varepsilon_2)E_3(4a)}$$
$$\times\int_{-a}^{a}[E_2(a-\tau')+(1-\varepsilon_1)E_2(3a+\tau')]I_0(\tau')\,d\tau'$$
$$+2\varepsilon_1 n^2\sigma T_1^4\left\{E_3(a+\tau)-\frac{2(1-\varepsilon_2)E_3(2a)}{1-2(1-\varepsilon_1)(1-\varepsilon_2)E_3(4a)}\times[E_3(a-\tau)-(1-\varepsilon_1)E_3(3a+\tau)]\right\}$$
$$-\frac{2\varepsilon_2 n^2\sigma T_2^4}{1-2(1-\varepsilon_1)(1-\varepsilon_2)E_3(4a)}\times[E_3(a-\tau)-(1-\varepsilon_1)E_3(3a+\tau)] \qquad (9)$$

Solution technique

The quantities of most outstanding physical relevance, q and $T(\tau)$, can be easily determined once $I_0(\tau)$ is known, so that one has to solve the integral equation (6) or (7). Projection methods have been chosen since convergence is guaranteed *a priori* and error estimates can be given for each approximation order. The Nth order approximate solution has then been represented as a linear combination of the first N elements of a suitable expansion basis, complete in $L_2(-a, a)$, with N unknown coefficients ξ_n^N. These coefficients have been determined by imposing the condition that the residual of the integral equation vanishes under the first N bounded linear functionals of a complete basis of the dual space, with two possible options. In the first option the functionals are those associated with the N expansion functions themselves (Bubnov–Galerkin method, equivalent to a Raleigh–Ritz variational approach), ie projection basis and expansion basis coincide. In the second option the projection functions are delta functions (point functionals) so that the actual requirement is vanishing of the residual at N selected points of the interval $(-a, a)$ (collocation method).

The basis that has been associated with the Bubnov–Galerkin method, for reasons of numerical stability, is the set of polynomials which is orthonormal in the range $(-a, a)$[13], namely the scaled Legendre polynomials

$$\left(\frac{2n-1}{2a}\right)^{1/2}P_{n-1}\left(\frac{\tau}{a}\right) \qquad (10a)$$

whereas a convenient choice for the collocation method is provided[14] by the set of Chebyshev polynomials of the first kind

$$C_{n-1}\left(\frac{\tau}{a}\right) \qquad (10b)$$

In the latter case the N collocation points (the nodes where the approximate integral equation is exactly satisfied) can be taken as the N simple zeros of the polynomial $C_N(\tau/a)$, explicitly given by

$$\cos\frac{(2m-1)\pi}{2N} \qquad m=1, 2, \ldots, N$$

The matrix elements of the linear algebraic system for the calculation of the expansion coefficients via the Bubnov–Galerkin method read as

$$\frac{(2m-1)^{1/2}(2n-1)^{1/2}}{2a}\int_{-a}^{a}P_{m-1}\left(\frac{\tau}{a}\right)d\tau \times\int_{-a}^{a}K(\tau,\tau')P_{n-1}\left(\frac{\tau'}{a}\right)d\tau' \qquad (11)$$

where $K(\tau, \tau')$ may be either $\frac{1}{2}E_1(|\tau-\tau'|)$, for Eq (6), or given by Eq (8a), for Eq (7). The double integral in Eq (11) involves integrations of polynomials and exponential integrals, and can be cast in closed analytical form in terms of elementary functions, incomplete gamma functions and, again, exponential integrals[9,10]. The same applies to the inhomogeneous terms of the same algebraic system

$$\left(\frac{2n-1}{2a}\right)^{1/2}\int_{-a}^{a}P_{n-1}\left(\frac{\tau}{a}\right)S(\tau)\,d\tau \qquad (12)$$

where now $S(\tau)$ is given either by $\frac{1}{2}E_2(a+\tau)$ or by Eq (8b).

The same quantities via collocation are formally simpler, since integrations over τ are replaced by specialization at suitable points, but the procedure becomes much more time-consuming since, contrary to the previous approach, all matrix elements and inhomogeneous terms must be re-computed at each approximation order, discarding all values of the preceding step.

The convergence rate is very good with both methods. The Bubnov–Galerkin method is better for low approximation orders, but collocation is less sensitive to numerical instabilities induced by the oscillating trend of polynomials when the approximation order is large. However, six converged significant figures for q are always achieved with approximation orders N not greater than 15 for optical thicknesses less than 50 mean free paths. (The thinner the slab, of course, the better is the polynomial approximation.) All layers of practical interest in radiant heat technology are actually very small in units of mean free paths (mfp). The procedures above become, then, particularly effective for accurate and inexpensive heat transfer calculations, since values of N of the order of 4 or 5 are sufficient for a 5 or 6 digits accuracy when $2a$ is smaller than 1 mfp, and the approximation $N=1$ already provides at least six correct digits when $2a<10^{-2}$ mfp. When $2a\rightarrow 0$ (no participating medium), q tends to the classical result

$$q_0=\frac{\sigma(T_1^4-T_2^4)}{\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}-1} \qquad (13)$$

Results

Solutions of the integral heat transfer equation, with either specular or diffuse reflection on the hot surface, have been carried out and radiative heat flux and temperature distribution in the participating medium have been computed and plotted. It is interesting to study the separate effects of the various physical parameters $2a$, ε_1, T_1 on the temperature distribution. The medium refractive index, the emissivity and temperature of the cold surface were kept constant throughout this analysis, with values $n=1$, $\varepsilon_2=0.8$, $T_2=400$ K.

The results for diffuse reflection from the hot surface are presented in Figs 1–3. Fig 1 shows the effect of the gap spacing $2a$, with $\varepsilon_1=0.2$ and $T_1=800$ K. As expected, the temperature distribution in the medium becomes steeper as the spacing increases, while the temperature slip at the boundaries decreases. For very

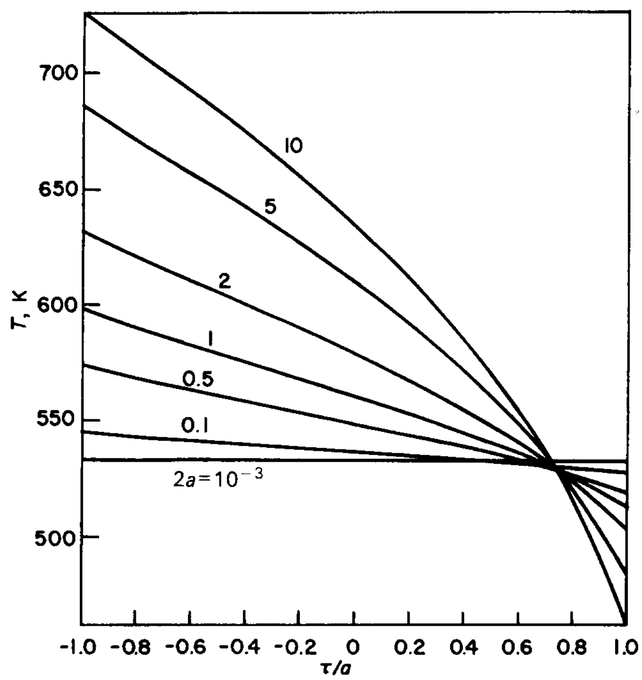

Fig 1 Temperature distribution in the medium for diffuse reflection from the hot surface, for varying gap spacing 2a ($\varepsilon_1=0.2$, $T_1=800$ K)

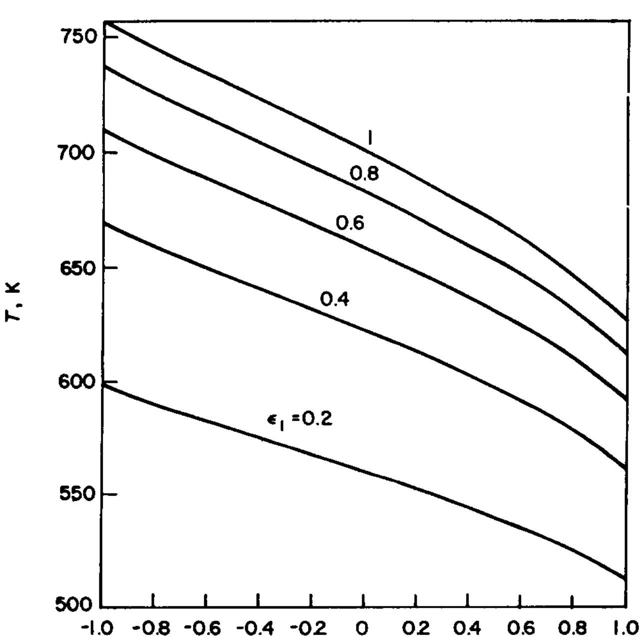

Fig 2 Temperature distribution in the medium for diffuse reflection from the hot surface, for varying ε_1 ($2a=1$ mfp, $T_1=800$ K)

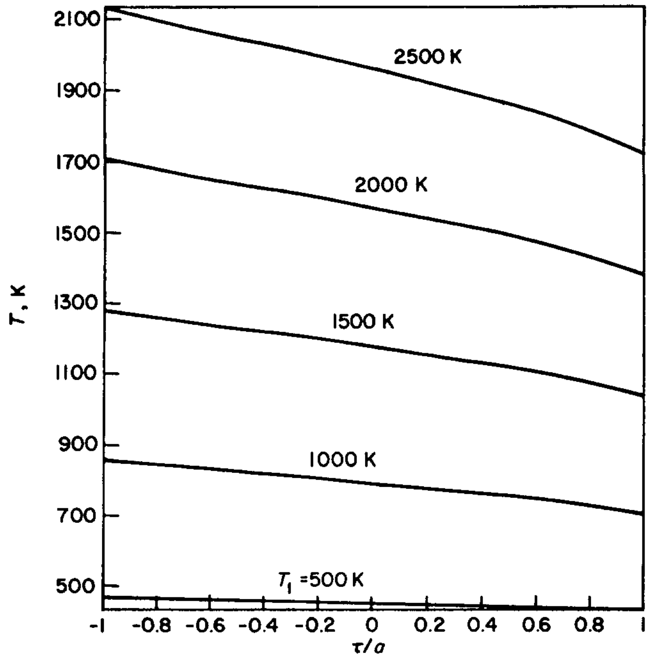

Fig 3 Temperature distribution in the medium for diffuse reflection from the hot surface, for varying T_1 ($2a=1$ mfp, $\varepsilon_1=0.5$)

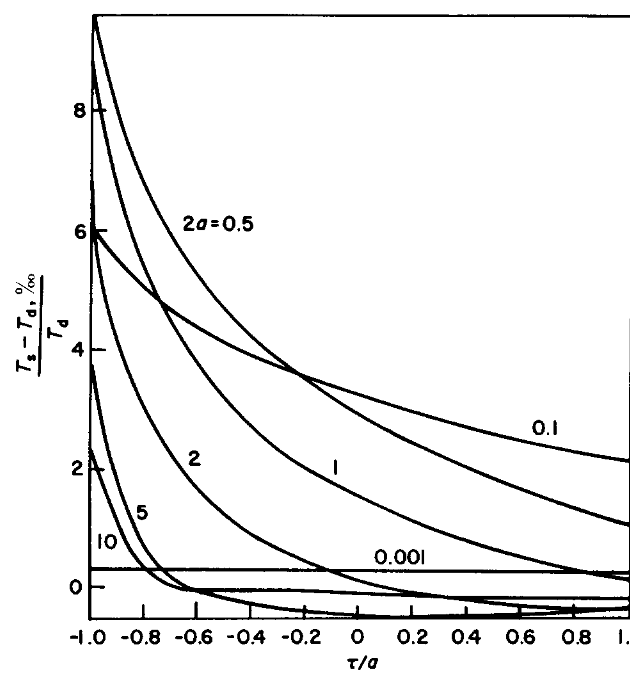

Fig 4 Relative temperature difference between specular and diffuse reflection from the hot surface, for varying 2a

thin layers ($2a<0.01$ mfp) the participating medium is approximately isothermal. The influence of the hot surface emissivity is shown in Fig 2 (with $2a=1$ mfp and $T_1=800$ K). The temperature level obviously increases with ε_1. The temperature distribution in the medium, for various values of T_1, is plotted in Fig. 3 (with $2a=1$ mfp and $\varepsilon_1=0.5$). The temperature slip on both boundaries increases with T_1.

A comparative analysis of the results obtained, considering separately specular and diffuse reflection by the hot surface, is given in Figs 4–6, where the relative difference $(T_{specular}-T_{diffuse})/T_{diffuse}$ is plotted versus τ/a for

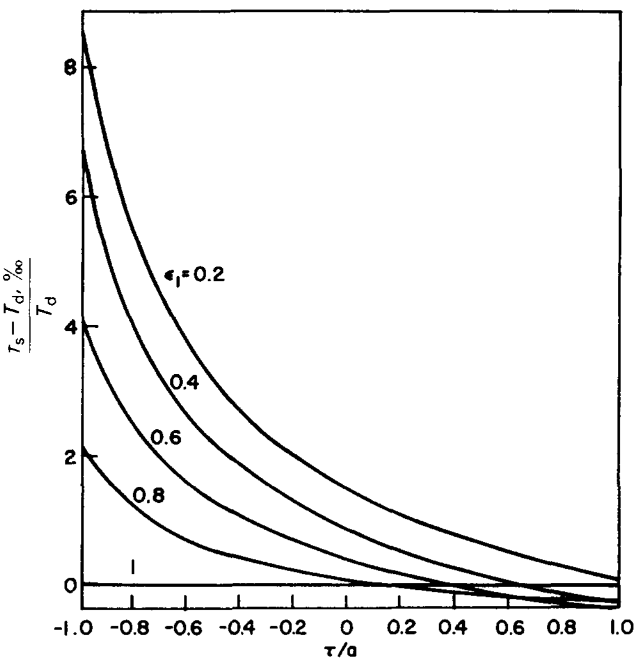

Fig. 5 Relative temperature difference between specular and diffuse reflection from the hot surface, for varying ε_1

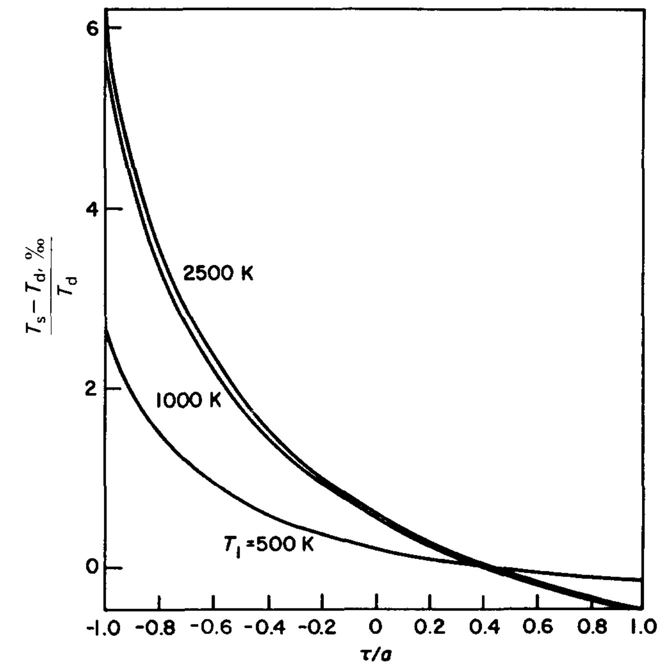

Fig 6 Relative temperature difference between specular and diffuse reflection from the hot surface, for varying T_1

Table 1 Heat flux (kW/m²) with diffuse and specular reflection from the hot surface, for various layer widths

	$2a \leq 10^{-3}$	$2a = 0.1$	$2a = 0.5$	$2a = 1$	$2a = 2$	$2a = 5$	$2a = 10$
Diffuse	4.146	4.075	3.839	3.594	3.195	2.401	1.699
Specular	4.146	4.073	3.829	3.581	3.182	2.393	1.691

Table 2 Heat flux in the vacuum and in the medium (with diffuse and specular reflection from the hot surface) for various emissivities of the hot surface

	$\varepsilon_1 = 0.2$	$\varepsilon_1 = 0.4$	$\varepsilon_1 = 0.6$	$\varepsilon_1 = 0.8$	$\varepsilon_1 = 1$
Vacuum	4.146	7.916	11.358	14.513	17.415
Diffuse	3.594	6.120	7.993	9.436	10.583
Specular	3.581	6.092	7.961	9.414	10.583

Table 3 Heat flux in the vacuum and in the medium (with diffuse and specular reflection from the hot surface) for various temperatures of the hot surface

	$T_1 = 500$ K	$T_1 = 1000$ K	$T_1 = 1500$ K	$T_1 = 2000$ K	$T_1 = 2500$ K
Vacuum	0.930	24.551	126.907	402.484	983.556
Diffuse	0.684	18.070	93.406	296.235	723.915
Specular	0.681	17.989	92.992	294.920	720.701

different values of $2a$, ε_1, T_1. The influence of $2a$ on the relative shifting is shown in Fig 4. For very thin or very large layers, this shifting (specular-diffuse) is very small, the greatest values (everywhere less than 1%) being relevant to intermediate thicknesses. The trends of the relative shifting are, of course, much more monotonic in Fig 5, where the influence of ε_1 is investigated. The shifting in fact increases as the emissivity of the hot surface decreases. For $\varepsilon_1 = 1$ (black body) there is no reflection, so the difference between the specular and diffuse temperature is zero. Fig 6 relates the same relative shifting to the hot surface temperature. The distributions for $1000\text{ K} \leq T_1 \leq 2500\text{ K}$ are nearly coincident. Obviously the highest shiftings always occur near the hot surface, where the reflection law is different, and this maximum increases for increasing T_1.

Tables 1 to 3 show the radiant net heat exchange, emphasizing again the separate effects of $2a$, ε_1, T_1; the flux is expressed in kW/m². In order to examine the effects of the participating medium, the heat flux in the vacuum (independent of the reflection mode on the boundaries) is also quoted. The two extreme cases of pure specular and pure diffuse reflection at the left edge are, as usual, reported. The radiative heat flux is always greater in the latter case than in the former, but the relative difference is quite small (less than 0.5%).

The conclusions that can be drawn from the Tables confirm those discussed for the graphs: in particular, the conspicuous effects of the participating medium (with respect to the vacuum) and the trifling difference between the cases with specular and diffuse reflection.

In conclusion, the two problems of diffuse and specular reflection at the hot boundary, solved quite separately, give very similar results for temperature and fluxes. The mathematical procedure and the computer solution are much less simple, and require a longer cpu time for the specular problem than for the diffuse one. For practical applications, where in general specular and diffuse reflections coexist, it can then be suitable to resort to the simplifying assumption of diffuse reflection only, in order to minimize the computer time without losing too much in terms of accuracy and reliability of the final results. For every specific problem, one should then examine, according to the accuracy required, whether it is worth facing more complicated models of specular, or combined diffuse–specular, reflection.

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